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ON CERTAIN CONSTRUCTIONS OF DESCRIPTIVE GEOMETRY.

EXTRACTS FROM LETTERS ADDRESSED BY

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to W. H. ROEVER, Washington University, St. Louis.

I.

It will doubtless not have escaped you that the month of July of this year marks the first centenary of the death of Gaspard Monge. The war has rendered impossible the execution of my project to publish at this time, as a monument to the memory of this immortal savant, my *History of Descriptive Geometry*, finished four years ago and intended to form the third volume of my *Vorlesungen über darstellende Geometrie*. Nevertheless my mind has recently been occupied —perhaps because of this centenary—with a group of questions, bearing on developments which have been influenced (or are likely to be influenced) by the ideas of this great mathematician, concerning that branch of geometry in which you are especially interested. Will you permit me therefore to share with you some of my thoughts on this subject?

Descriptive geometry had birth in the mind of Monge as an auxiliary to the engineer, destined (as he says in the beginning of his work) "to draw the French nation away from the dependence which it has had up to this time on foreign industry." He concerned himself then with a science, which certainly in an embryonic form, is found among all civilized peoples. But while architects and painters desire in general to represent objects as they appear, Monge has given a process for performing (by drawings in a plane) all sorts of geometric operations on figures of three dimensions. Hence one may say, that, while the method of double (orthographic) projection as used, for example, by Vitruvius in his celebrated work on architecture is *static* in character, with Monge it becomes *dynamic.* Consequently, this method, which in the first stages of its existence served almost exclusively the artist, was elevated by Monge to the level of geometric research. And in effect the disciples of Monge will often unconsciously have recourse to this method of research.

In order to justify this last statement I am going to cite a noteworthy example. In the course of his important studies on the reflection of light, A. Quetelet established the following remarkable theorem: "Let us conceive two right cones having their vertices in different points and their axes parallel to each other; the intersection of these cones projected on a plane perpendicular to their axes will give aplanetic lines (ovals of Descartes)." From this proposition follow two methods of describing these curves; the one is the application of the method of construction which serves to find the projections of the curve of intersection of two cones represented by the method of Monge[1] while the other is an application of the general method of construction for finding the curve of intersection of two surfaces of revolution with parallel axes.[2] That portion of the first of these constructions which yields the horizontal projection can be enunciated as follows: "Given two circles in a plane; if around a fixed point taken on the line which joins their centers a transversal be made to turn cutting the two circles each in two points, the rays from the centers of the circles to their respective points of intersection with the transversal will meet in four points of which the geometric locus will be a complete aplanetic having its two foci situated at the centers of the two circles." Now this enunciation can be found in *Note* XXI of *l'Aperçu historique* of M. Chasles[3] where it is presented as a corollary of "the theorem of Ptolemy on the triangle cut by a transversal." But it is my opinion that the great French geometer reached this conclusion by combining the theorem of Quetelet, which I have quoted above and which he cited at the beginning of this note, with what Monge had taught him at l'Ecole polytechnique. If he has preferred to reach this conclusion by the theory of transversals it is for the reason that thereby he avoids geometry of space and thus uses methods characteristic to himself.

This example (to which I will return later), shows that the method of double projection can be very successfully used in questions relative to the construction of plane curves. However, it is not the only example which leads one to conclude that *the methods of descriptive geometry should be among the ordinary tools of those who cultivate the geometry of plane curves.* Such a recommendation may appear superfluous since descriptive geometry has become, among the majority of civilized peoples, an integral part of the program of studies for candidates of the doctor's degree in mathematics; but I am going to cite a fact which appears to lead to a different conclusion.

Among the most ancient curves known one finds "les spiriques de Perseus"; they are the sections cut from a torus (annular surface) by a plane parallel to the axis of the surface. In order to describe them "in plano" R. de Sluse in

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[1] See my *Vorlesungen* already cited, Vol. II, p. 205.

[2] *Vorlesungen*, Vol. II, p. 262.

[3] II ed., Paris, 1875, p. 351.

1637 devised a somewhat complicated construction in which an hyperbola was employed. In 1905 F. Gomese Teixeira indicated a method which has the advantage of requiring only the use of straight lines and circles. However, the simplest and most natural method is the classic one of constructing the curve of intersection of a plane with a surface of revolution of vertical axis. When the method of double projection is used to represent the surfaces in question and the second plane of projection is taken to be parallel to the cutting plane,[1] then the vertical projection of the curve becomes identical with the objective curve, and the end sought is attained by a solution the details of which can be found in a letter which I addressed to M. Teixeira and which he published in Vol. IX (pp. 193–196) of the *Annaes scientificos da Academia polytechnica do Porto*. The thing is so simple and natural that I could never persuade myself to make it the subject of a publication as such. However, it appears that no one had thought of it before I did. If it did not concern itself with such a little thing, one might see in all this a confirmation of the observation of Jacobi that discoveries are born of the contact of theories which were at first separate.

Permit me to return to the construction of the ovals of Descartes discovered by Quetelet. These curves, as you know, possess three real foci which are situated on a straight line. But one can consider curves entirely analogous to these of which one focus alone is real while the other two are conjugate imaginaries. A. Cayley was the first to consider these curves in his "Note on the cartesian with two imaginary axial foci." [2] He called them Cartesian lines and established some of their properties, but he did not arrive at a simple construction nor did he determine their form. Now all of these things can be obtained by a consideration of geometry of three dimensions because there exists a simple relation between these curves and the ovals of Descartes, which have already been considered in a preceding paragraph.

A cartesian curve may be obtained as the orthographic projection of the curve of intersection of two cones of revolution which are conjugate imaginaries and of which the axes are parallel to each other, provided the plane of projection is normal to the axes. As descriptive geometry concerns itself exclusively with real elements, this remark does not lead at once to the desired result; but it is natural to ask whether it is not possible to substitute for the two imaginary cones two real surfaces of the pencil of surfaces which they determine. Now I have recently demonstrated in the note "Fasci di quàdrichi rotonde e Curve cartesiane," [3] of which I have the honor to send you a copy, that "through the intersection of two real or conjugate imaginary cones of revolution of which the axes are parallel, there passes a third cone of revolution of which the axis is parallel to the axes of the given cones and also a sphere, both of these surfaces being real; if the two given cones are real, the intersection is composed of two branches and its projection on the plane perpendicular to the axes is a pair of conjugate ovals of

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[1] See my *Vorlesungen*, Vol. II, p. 258.

[2] *Proc. of the London Math. Society*, Vol. III, 1871–72 or also *The Collected Papers*, Vol. VII, pp. 241–243.

[3] *Rend. della R. Accademia dei Lincei*, sèance du 17 Mars, 1918.

Descartes; but if the given cones are conjugate imaginaries, the intersection has only one branch and then its projection is a cartesian line." It is thus demonstrated that the ovals of Descartes and the cartesian curves may be obtained in the same manner, namely, as the orthographic projection of the curve of intersection of a real sphere by a real cone of revolution, and that the first or second type of curve is obtained according as the cone actually penetrates or merely "bites" the sphere. Now since a sphere and a cone of revolution may be regarded as two surfaces of revolution with parallel axes, the orthographic projection of their intersection may be obtained by the aid of a classic construction of descriptive geometry to which I have already had occasion to refer. In Fig. 1 I have carried through this construction, supposing, as is permissible, that the vertical plane of projection is parallel to the plane determined by the axis of the cone and the center of the sphere. In the figure the horizontal projection only of the curve of intersection is constructed, and thus I have arrived at the *first construction which I know of cartesian lines.*

I will finish with the remark that that which precedes enables us to complete another theorem of Quetelet. In effect, this geometer demonstrated that if one has a sphere and a right cone and makes a stereographic projection of the curve of penetration of these surfaces, the eye being placed at an extremity of the diameter of the sphere which is parallel to the axis of the cone and the plane of projection taken perpendicular to this axis, then the projection will be a pair of conjugate ovals of Descartes. Now, if the cone instead of penetrating the sphere, merely "bites" it, the projection will be a cartesian line.

II.

The application of considerations of geometry of space to geometry of the plane is one of the most brilliant ideas of Monge. But it is remarkable that in the hands of some of his pupils (I cite particularly Hachette and Olivier) it has undergone a strange and vexatious deviation to which I wish to call your attention.

One of the principal aims of descriptive geometry is to replace by drawings in a plane the purely ideal constructions of geometry of three dimensions. Until some physicist will have taught us how to *design in space,* the inverse process, namely that of replacing a planometric construction by a stereometric one, represents from the graphic point of view a veritable step backward. Nevertheless, in several works of the two geometers whom I have mentioned, one finds space constructions which have for their object the determination of tangents to particular curves—constructions which give the deplorable delusion of having solved problems which, on the contrary, are yet unsolved. I think, on the contrary, that descriptive geometry should have as a basis a complete and detailed knowledge of all of plane geometry. It is for this reason that I long for the publication of a comprehensive work of which Part I shall comprise all constructive geometry of plane figures, while Part II shall be a treatise on descriptive geometry. In order to render more striking the utility of such a fusion, I will cite an example.

In 1826, H. Garbinski, the apostle of descriptive geometry in Poland, published in the *Annales de Mathématiques* de Gergonne (T. XVI, pp. 167–172), an article on the conical spiral, a curve which he conceived in generalizing the definition of the ordinary cylindrical helix, but which B. Pascal had considered two centuries before. This curve is the trajectory described by a point, which, starting at the vertex of a right circular cone, describes with uniform motion a generatrix, which in turn moves around the surface of the cone with a constant velocity. From this definition the following properties may be deduced: (1) The orthographic projection of the conical spiral on a plane normal to the axis of the cone is a spiral of Archimedes. (2) The conical spiral lies on the surface of a right helicoid, which is coaxial with the cone of this spiral. In order to represent the curve in question by the method of Monge, one can use a process which is entirely analogous to that which is used to represent the cylindrical helix (see Fig. 2). The curve may be represented analytically by the following equations:

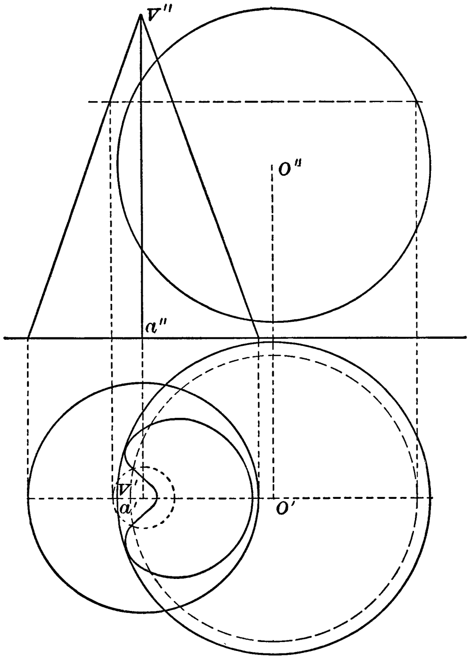

FIG. 1.

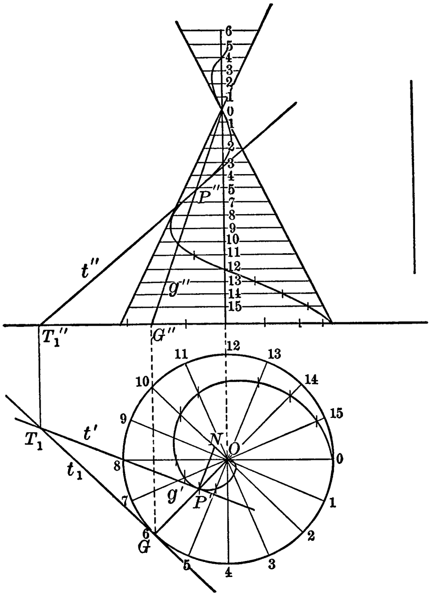

FIG. 2.

$$(1) \qquad x = \frac{r\omega}{2\pi}\cos\omega, \qquad y = \frac{r\omega}{2\pi}\sin\omega, \qquad z = l - \frac{l\omega}{2\pi},$$

in which l is the height of the cone, r is the radius of its base, and ω is a parameter. These equations prove that the orthographic projection Γ' of the spiral Γ on the plane xy is the spiral of Archimedes

$$(2) \qquad \rho = \frac{r\omega}{2\pi}$$

and that Γ lies upon the surface of the right helicoid

$$\frac{y}{x} + \tan\frac{2\pi z}{l} = 0. \tag{3}$$

Equation (2), on differentiation, yields the equation

$$\frac{d\rho}{d\omega} = \frac{r}{2\pi}, \tag{4}$$

and consequently the curve Γ' enjoys the remarkable property that its polar subnormal ON in any point whatever P' has the constant value $r/2\pi$. It is then, indeed, easy to construct the normal, and consequently the tangent t' to Γ' at the arbitrary point P', that is to say, the horizontal projection of the tangent t at a general point of the conical spiral. To complete the representation of the tangent t, let us remark (1) that t'' passes through P'', (2) that t lies in the plane τ which is tangent to the given cone along the generatrix g which passes through the point P. Now if G is the horizontal trace of g, the horizontal trace of the plane τ is the tangent t_1 at the point G to the circular base of the given cone. Then the intersection T_1 of t' and t_1 is the horizontal trace of the required tangent, while t'' is the line which joins T_1'' and P''.

Now in place of having recourse to this simple solution, Garbinski, whose knowledge of the spiral of Archimedes seemed rather limited, made use of the fact that the curve Γ lies on the helicoid (3), and therefore that the tangent sought lies in the corresponding tangent plane to this surface. He thus constructed this plane by the aid of a related hyperbolic paraboloid. From another point of view this last artifice was not indispensable, for as Vallès remarked at once,[1] it is most simple to make use of one of the cylindrical helices lying on the helicoid. As a consequence of all of these considerations, Garbinski and Vallès gave constructions for the tangent to the spiral of Archimedes which were extremely more complicated than the construction which is a result of the constancy of the subnormal.

I wish to add that the preceding construction of the tangent to the conical spiral may serve also an analogous end for the *conical helix* (that is the curve which is now called, on account of one of its remarkable properties, the helical cylindrical-conic). This curve has the property of projecting itself orthographically on the plane of the base of the cone into a logarithmic spiral. Now the tangents of this curve make constant angles with the corresponding radii vectores. Consequently nothing is easier than to construct the horizontal projection of the tangent to the conical helix. Then the vertical projection is obtained by the aid of the tangent plane to the given cone precisely as in the case of the conical spiral.

III.

Although descriptive geometry may have been conceived by Monge as a practical science, I have not been able to find in the publications of this great

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[1] *Annales de Mathématiques*, T. XVI, pp. 376–377.

geometer any word relative to the necessity for obtaining the *most simple solutions possible,* that is to say, those that are at the same time the most economical and the most exact. The criterion of simplicity for the choice among several solutions of the same problem is entirely modern; to be sure it has not yet been applied to its full extent, with the result that researches in this direction make it possible, if I do not deceive myself, to penetrate farther into several domains which are now regarded by some writers as having been exhaustively studied. To prove to you that I do not allow myself to be deceived, I am going to return to a classic problem, namely, to the construction of the tangent t at a general point P (Fig. 4) of the cylindrical helix, ordinarily represented by the method of Monge. It is known[1] that to effect this construction one begins by drawing the horizontal projectiont' of the tangent t (that is, the tangent t' at the point P' of the circular base of the given cylinder). After determining the horizontal trace T_1 of t, the second projection t'' of t is found by connecting T_1'' with P''. But in order to determine T_1, it is necessary to find a rectilinear segment equal in length to the arc of the circular base which is comprised between P' and the point where the helix pierces the horizontal plane. Now the rectification of a circular arc is a very delicate transcendental operation which may be performed by a simple approximation process,[2] and which must be repeated for each point of the curve where the tangent is required. Consequently I flatter myself that I may interest you in a process which holds for all points of the helix—a unique transcendental operation which is more simple than the rectification of the circular arc. In order to establish this result, I will make use of the following remark (to which, moreover, we will have recourse in several analogous questions): "If one considers two projections Γ' and Γ_0 (parallel or central) of a curve Γ on the same plane σ, the tangents to these curves at any two corresponding points P' and P_0 will intersect in the trace on the plane σ, of the tangent at the point P, to the objective curve Γ." To apply it to our question, let us recall[3] that if one projects the helix, represented by the equations

$$x = r \cos \varphi, \qquad y = r \sin \varphi, \qquad z = \frac{H}{2\pi} \varphi, \tag{5}$$

from the point $(0, 0, c)$ on the plane xy, there results the hyperbolic spiral having for its polar equation

$$\rho \left(\frac{2\pi c}{H} - \omega \right) = \frac{2\pi c r}{H}, \tag{6}$$

from which by differentiation one obtains the equation,

$$\frac{1}{\rho} \frac{d\rho}{d\omega} \left(\frac{2\pi c}{H} - \omega \right) = 1. \tag{7}$$

On dividing member by member equation (6) by equation (7), one obtains the relation

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[1] See *Vorlesungen,* Vol. II, p. 137.

[2] See *Vorlesungen,* Vol. II, p. 63.

[3] *Vorlesungen,* Vol. II, pp. 130–32.

$$\rho^2\frac{d\omega}{d\rho}=\frac{2\pi cr}{H} \tag{8}$$

which shows that the curve (6) possesses the property that its polar subtangent has a constant value $l = 2\pi cr/H$. In order to determine a segment equal in length to l (Fig. 3), let us construct by the method of Kochanski,[1] or by any other process, a line AD equal to the half circumference πr of the circle of diameter $AB = 2r$. Let us now lay off on AB the segments $AM = H/2$ and $AN = c$, and then join M to D and draw NX parallel to MD. Then

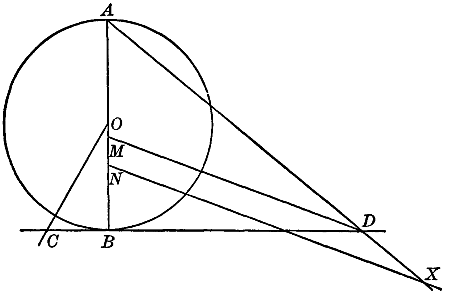

FIG. 3.

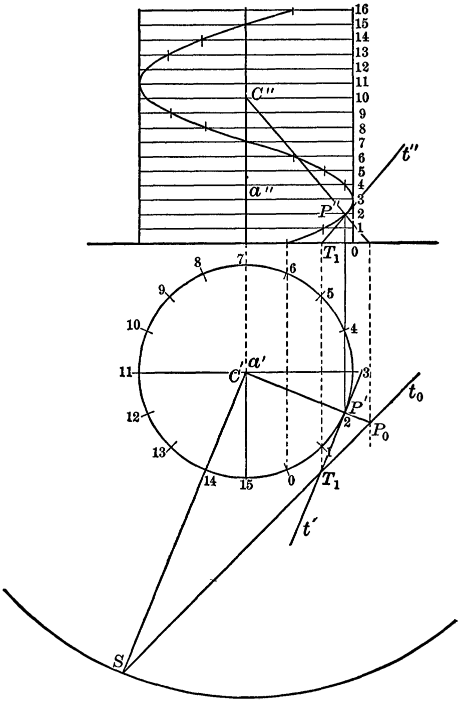

FIG. 4.

$$\frac{AX}{AN}=\frac{AD}{AM},\quad\text{or indeed}\quad\frac{AX}{c}=\frac{\pi r}{H/2},\quad\text{whence}\quad AX=\frac{2\pi cr}{H}=l.$$

We may now pass to Fig. 4, which represents the given helix. Let $P \equiv (P', P'')$ be any point of the helix and P_0 its projection from the center $C \equiv (C', C'')$ on the horizontal plane. Describe the circle of center a' and radius l and draw in this circle the radius $a'S$ making with the line $a'P'$ an angle of 90° in the negative sense of rotation. Then the line SP_0 will be the tangent at the point P_0 to the hyperbolic spiral, that is to say, the central projection, from the center

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[1] *Vorlesungen*, Vol. II, p. 63.

C, of the tangent, at the point P, to the given helix. This tangent cuts the tangent t', which is drawn at the point P' to the circle Γ, in the horizontal trace T_1 of the required tangent. Etc.

GENOA, May–June, 1918.

DIFFERENCE QUOTIENTS.

By J. P. BALLANTINE, Cambridge, Mass.

The whole subject of interpolation, in the case of equally spaced ordinates, by means of algebraic polynomials is commonly based on the subject of finite differences. The method, however, fails for unequal spacing. The purpose of this paper is to show, that by a suitable extension of the conception of the difference quotient, the method can be generalized for the case when the ordinates are spaced according to any law.[1]

Let $f(x)$ be any single valued function. Plot the curve $y = f(x)$. Pick out on the curve the $(n+1)$ points whose x coördinates are $1, 2, \cdots, (n+1)$, respectively. By means of the method of finite differences, find the equation of the curve of the form $y = P_n(x)$ which passes through the $(n+1)$ points. For convenience, call this curve a secant of degree n, noticing that a secant of the first degree is an ordinary secant. It is easily verified that the nth derivative of this secant is precisely equal to the difference of order n of the ordinates of the $(n+1)$ points.

Now let us take our $(n+1)$ points on the curve at random. We have to define the difference quotient of order n at these $(n+1)$ points in such a way that when the values of $x_1, x_2, \cdots, x_{n+1}$ are $1, 2, \cdots, n+1$ respectively, it will reduce to a difference of order n.

We can define the secant through these $(n+1)$ points as the curve of the form $y = P_n(x)$ which passes through them. By way of a more explicit notation, $y = P_n(x;\ 1, n+1)$ will be the equation of the nth degree secant at the points $(x_1, y_1), (x_2, y_2)\cdots(x_{n+1}, y_{n+1})$. It follows that

$$P_n(x_i;\ 1, n+1) = y_i, \qquad 1 \leqq i \leqq n+1.$$

DEFINITION. The difference quotient of order n, taken at the $(n+1)$ points $(x_1, y_1), (x_2, y_2)\cdots(x_{n+1}, y_{n+1})$ is the nth derivative of the nth degree secant through those points. It is denoted thus $Q^n_{1, n+1}$:

$$Q^n_{1, n+1} = \frac{d^n}{dx^n} P_n(x;\ 1, n+1).$$

Consider the special polynomial and associated equation

$$P_{n+1}(x;\ 1, n+2) - P_n(x;\ 1, n+1),$$

$$(1) \quad P_{n+1}(x_i;\ 1, n+2) - P_n(x_i;\ 1, n+1) = y_i - y_i = 0,\ 1 \leqq i \leqq n+1.$$

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[1] Cf. A. A. Markoff, *Differenzenrechnung*, 1896, p. 10.